DISCRETE SECOND-ORDER EULER-POINCARÉ EQUATIONS. APPLICATIONS TO OPTIMAL CONTROL

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Abstract. In this paper we will discuss some new developments in the design of numerical methods for optimal control problems of Lagrangian systems on Lie groups. We will construct these geometric integrators using discrete variational calculus on Lie groups, deriving a discrete version of the second-order Euler-Lagrange equations. Interesting applications as, for instance, a discrete derivation of the Euler-Poincaré equations for second-order Lagrangians and its application to optimal control of a rigid body, and of a Cosserat rod are shown at the end of the paper.

1. Introduction

The goal of this paper is to study, from a geometric point of view, variational integrators for optimal control problems of mechanical systems defined on finite dimensional Lie groups, and its applications in optimal control theory. Our motivation is the control of autonomous vehicles modeled as rigid bodies (as an evolution equation in time).

We use the theory of discrete mechanics based on discrete variational calculus [25]. In particular, we use Hamilton's principle yielding the set of discrete paths that approximately satisfy the dynamics. This is achieved by formulating a second order discrete variational problem solved through discrete Hamilton's principle on Lie groups and obtaining a variational numeric integrator. Such formulation gives us the preservation of important geometric properties of the mechanical system, such as momentum, symplecticity, group structure, good behavior of the energy, etc [11].

A typical optimal control problem consists on finding a trajectory of the state variables and controls $(g(t), \xi(t), u(t))$ given fixed initial and final conditions $(g(0), \xi(0))$ and $(g(T), \xi(T))$ respectively, and, as well, minimizing the cost functional defined by

$$J(u,T) = \int_0^T \|u(t)\|^2 dt;$$

here, $g(t)$ evolves on a Lie group G, $\xi(t)$ on the associated Lie algebra $\mathfrak{g}$ and $u(t)$ on the space of admissible controls.

Our approach is based on recently developed structure-preserving numerics integrators for optimal control problems (see [8],[9],[16], [17], [20], [27] and references therein) based on solving a discrete optimal control problem as a discrete higher-order variational problem with higher-order constraints (see [3] for the continuous case) which are used for simulating and controlling

the dynamics for satellites, spacecrafts, underwater vehicles, mobile robots, helicopters, wheeled vehicles, mobile robots, etc [5].

1.1. **Background: Discrete Mechanics and variational integrators.** Let Q be a n-dimensional differentiable manifold, the configuration manifold, with local coordinates (q^i), $1 \leq i \leq n$. Denote by TQ its tangent bundle with induced coordinates $(q^i, \dot{q}^i)$. Given a Lagrangian function $L : TQ \to \mathbb{R}$, the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \leq i \leq n. \tag{1}$$

These equations are a system of implicit second order differential equations.

In the sequel, we will assume that the Lagrangian is **regular**, that is, the matrix $\left(\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ is non-singular. It is well known that the origin of these equations is variational (see [1],[24]).

Variational integrators [25] are derived from a discrete variational principle. These integrators also retain some of main geometric properties of the continuous system, such as simplecticity, momentum conservation and a good behavior of the energy associated with the Lagrangian system (see [11] and references therein).

In the sequel we will review the construction of this type of geometric integrators.

A **discrete Lagrangian** is a map $L_d \colon Q \times Q \to \mathbb{R}$, which may be considered as an approximation of the integral action defined by a continuous Lagrangian $L \colon TQ \to \mathbb{R}$,

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))\, dt$$

where $q(t)$ is a solution of the Euler-Lagrange equations for L; $q(0) = q_0$, $q(h) = q_1$ and the time step $h > 0$ is small enough.

Define the **action sum** $\mathcal{A}_d \colon Q^{N+1} \to \mathbb{R}$, corresponding to the Lagrangian L_d by

$$\mathcal{A}_d = \sum_{k=1}^{N} L_d(q_{k-1}, q_k),$$

where $q_k \in Q$ for $0 \leq k \leq N$, where N is the number of steps. The discrete variational principle then requires that $\delta \mathcal{A}_d = 0$ where the variations are taken with respect to each point q_k, $1 \leq k \leq N-1$ along the path, and the resulting equations of motion (system of difference equations) given fixed endpoints q_0 and q_N, are

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0, \tag{2}$$

where D_1 and D_2 denote the derivative to the Lagrangian respect the first and second arguments, respectively.

These equations are usually called **discrete Euler–Lagrange equations**. Under some regularity hypotheses (the matrix $(D_{12} L_d(q_k, q_{k+1}))$ is regular), it is possible to define a (local) discrete flow $\Upsilon_{L_d} \colon Q \times Q \to Q \times Q$, by $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$ from (2).

We introduce now the two discrete Legendre transformations associated to L_d:

$$\begin{array}{rcl} \mathbb{F}^{-}L_d : Q\times Q & \to & T^*Q \\ (q_0,q_1) & \mapsto & (q_0, -D_1L_d\,(q_0,q_1))\,, \\ & & \\ \mathbb{F}^{+}L_d : Q\times Q & \to & T^*Q \\ (q_0,q_1) & \mapsto & (q_1, D_2L_d\,(q_0,q_1))\,, \end{array} \tag{3}$$

and the discrete Poincaré-Cartan 2-form $\omega_d = \left(\mathbb{F}^{+}L_d\right)^*\omega_Q = \left(\mathbb{F}^{-}L_d\right)^*\omega_Q$, where ω_Q is the canonical symplectic form on T^*Q. ω_d is a symplectic form if the discrete Lagrangian is regular, which is indeed equivalent to $\mathbb{F}^{-}L_d$ (or $\mathbb{F}^{+}L_d$) being a local diffeomorphism.

The discrete algorithm determined by Υ_{L_d} preserves the (pre-)symplectic form on $T^*(Q\times Q)$, ω_d, i.e., $\Upsilon^*_{L_d}\omega_d = \omega_d$. Moreover, if the discrete Lagrangian is invariant under the diagonal action of a Lie group G, then the discrete momentum map $J_d\colon Q\times Q\to\mathfrak{g}^*$ defined by

$$\langle J_d(q_k,q_{k+1}),\xi\rangle = \langle D_2L_d(q_k,q_{k+1}),\xi_Q(q_{k+1})\rangle$$

is preserved by the discrete flow. Therefore, these integrators are symplectic-momentum preserving. Here, ξ_Q denotes the fundamental vector field determined by $\xi\in\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G,

$$\xi_Q(q) = \frac{d}{dt}\Big|_{t=0}(exp(t\xi)\cdot q)$$

for $q\in Q$ (see [25] for more details).

Example 1.1. For instance we consider a Lagrangian $L(q,\dot q) = \frac{1}{2}\dot q^T M\dot q - V(q)$, where $q\in\mathbb{R}^3$, M being a symmetric non-degenerate matrix and V a potential function. From this Lagrangian we construct the discrete Lagrangian taking an Euler's discretization,

$$L_d(q_k,q_{k+1}) = h\left[\left(\frac{q_{k+1}-q_k}{h}\right)^T M\left(\frac{q_{k+1}-q_k}{h}\right) - V(q_k)\right].$$

We compute $D_1\,L_d$ and $D_2\,L_d$:

$$\begin{array}{rcl} D_1L_d(q_k,q_{k+1}) & = & -M\left(\dfrac{q_{k+1}-q_k}{h}\right) - h\nabla V(q_k), \\ D_2L_d(q_{k-1},q_k) & = & M\left(\dfrac{q_k-q_{k-1}}{h}\right), \end{array}$$

which leads to the discrete Euler-Lagrange equations:

$$M\left(\frac{q_{k+1}-2q_k+q_{k-1}}{h^2}\right) = -\nabla V(q_k).$$

We observe that these equations give rise a natural discrete version of the Newton's law $M\ddot q = -\nabla V(q)$, using a simple finite difference rule for the derivative (see [25]).

1.2. **Organization of the paper.** The paper is structured as follows. In Section 2 we recall some results given in [23] about Hamilton's principle on Lie groups and the discrete Euler-Poincaré equations. The new proposed method appears in Section 3. First, we derive the continuous second-order Euler-Poincaré equations on Lie groups from Hamilton's principle; next, we construct from a discretization of the Lagrangian and through discrete variational calculus the discrete second-order Euler-Lagrange and Euler-Poincaré equations. The discrete higher-order Euler-Lagrange and discrete higher order Euler-Poincaré equations are derived using discrete Hamilton's principle. In the last section, we apply these techniques to optimal control of mechanical systems and we analyze two examples of optimal control on a rigid body on the Lie group $SO(3)$ and on a Cosserat rod defined on $SE(3)$.

2. Discrete mechanics on Lie groups

In this section we recall the discrete mechanics on Lie groups and Hamilton's principle on Lie groups for the formulation of Euler-Poincaré equations.

2.1. **Discrete Hamilton's principle on Lie groups and Euler-Poincaré equations.** If the configuration space is a Lie group G, then the discrete trajectory is represented numerically using a set of $N+1$ points $(g_0, g_1, \ldots, g_N)$ with $g_i \in G$, $0 \leq i \leq N$.

A way to discretize a continuous problem is using a *retraction map* $\tau : \mathfrak{g} \to G$ which is an analytic local diffeomorphism which maps a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the neutral element $e \in G$. As a consequence, it is possible to deduce that $\tau(\xi)\tau(-\xi) = e$ for all $\xi \in \mathfrak{g}$. The retraction map is used to express small discrete changes in the group configuration through unique Lie algebra elements (see [17]), namely $\xi_k = \tau^{-1}(g_k^{-1} g_{k+1})/h$, where $\xi_k \in \mathfrak{g}$. That is, if ξ_k were regarded as an average velocity between g_k and g_{k+1}, then τ is an approximation to the integral flow of the dynamics. The difference $g_k^{-1}\, g_{k+1} \in G$, which is an element of a nonlinear space, can now be represented by the vector ξ_k, in order to enable unconstrained optimization in the linear space $\mathfrak{g}$ for optimal control purposes.

It will be useful in the sequel, mainly in the derivation of the discrete equations of motion, to define the *right trivialized* tangent retraction map as

$$T_\xi\, \tau = T_e r_{\tau_\xi} \circ \mathrm{d}\tau_\xi.$$

Useful and complementary definition of the right trivialized (and its inverse) is the following ([13], [4]):

Proposition 2.1. Given a map $\tau : \mathfrak{g} \to G$, its right trivialized tangent $\mathrm{d}\tau_\xi : \mathfrak{g} \to \mathfrak{g}$ and its inverse $\mathrm{d}\tau_\xi^{-1} : \mathfrak{g} \to \mathfrak{g}$, are such that for $g = \tau(\xi) \in G$ and $\eta \in \mathfrak{g}$, the following holds

$$\partial_\xi \tau(\xi)\, \eta = d\tau_\xi\, \eta\, \tau(\xi),$$
$$\partial_\xi \tau^{-1}(g)\, \eta = d\tau_\xi^{-1}(\eta\, \tau(-\xi)).$$

An example of retraction map is the exponential map at the identity e of the group G, $exp_e : \mathfrak{g} \to G$. We recall that for a finite dimensional Lie grup, exp_e is locally a diffeomorphism and gives rise a natural chart [23]. Then, there exists a neighborhood U of $e \in G$ such that $exp_e^{-1} : U \to exp_e^{-1}(U)$ is

a local $\mathcal{C}^\infty$ diffeomorphism. A chart at $g \in G$ is given by $\Psi_g = exp_e^{-1} \circ l_{g^{-1}}$, where l denote the left-translation of an element of the group.

In general, it is not easy to work with the exponential. For instance, if we are considering matrix groups, the right trivialized derivative and its inverse are defined by infinite series

$$\begin{aligned} \mathrm{dexp}_x\, y &= \sum_{j=0}^{\infty} \frac{1}{(j+1)!}\, \mathrm{ad}_x^j\, y, \\ \mathrm{dexp}_x^{-1}\, y &= \sum_{j=0}^{\infty} \frac{B_j}{j!}\, \mathrm{ad}_x^j\, y, \end{aligned}$$

where B_j are the Bernoulli numbers, $x, y \in \mathfrak{g}$ and $\mathrm{ad}_x\, y = [x, y]$ is the usual matrix bracket (see [11]). Tipically, these expressions are truncated in order to achieve a desired order of accuracy.

In consequence it will be useful to use a different retraction map. More concretely, the Cayley map (see [11] for further details) will provide us a proper framework in the examples shown below.

The following theorem, regardless of the *retraction structure* locally relating G and $\mathfrak{g}$, gives us the relation between the discrete Euler-Lagrange equations and the discrete Euler-Poincaré equations.

Theorem 2.1. [?] *Let G be a Lie group and $L_d : G \times G \to \mathbb{R}$ a discrete Lagrangian function. We suppose that L_d is left-invariant over the diagonal action (i.e; $L_d(gg_k, gg_{k+1}) = L_d(g_k, g_{k+1})$ with $g \in G$). Let $\tilde{l}_d : G \to \mathbb{R}$ be the restriction to the identity (that is, $\tilde{l}_d : (G \times G)/G \simeq G \to \mathbb{R}$, $\tilde{l}_d(g_k^{-1} g_{k+1}) = L_d(g_k, g_{k+1})$). For a pair of points $(g_k, g_{k+1}) \in G \times G$, we consider $W_k = g_k^{-1} g_{k+1}$ (where $g_k^{-1} = i(g_k), i : G \to G$ the inversion map of the Lie group G). Then the following assertions are equivalent:*

1. *$(g_k)_{0 \le k \le N}$ satisfies the discrete Euler-Lagrange equations for L_d.*
2. *$(g_k)_{0 \le k \le N}$ extremize the discrete action*

$$(g_k)_{0 \le k \le N} \mapsto \sum_{k=0}^{N-1} L_d(g_k, g_{k+1})$$

   for all variation with initial and final fixed points.

3. *The discrete Euler-Poincaré equations*

$$r_{W_k}^* \tilde{l}'_d(W_k) - l_{W_{k-1}}^* \tilde{l}'_d(W_{k-1}) = 0 \qquad k = 1, \ldots, N$$

   hold, where l and r are the left- and right-translation of the Lie group and $'$ denote the partial derivative.

4. *$(W_k)_{0 \le k \le N-1}$ extremize*

$$(W_k)_{0 \le k \le N-1} \mapsto \sum_{k=0}^{N-1} \tilde{l}_d(W_k)$$

   for all variations $\delta W_k = -\Sigma_k W_k + W_k \Sigma_{k+1}$ with $\Sigma_0 = \Sigma_N = 0$; where $\Sigma_k \in \mathfrak{g}$ is given by $\Sigma_k = g_k \delta g_k$.

3. Continuous and discrete Euler-Poincaré equations for second order lagrangians

In this section we derive, from a variational point of view, the discrete and continuous Euler-Lagrange equations for second-order Lagrangians defined on Lie groups: the second order Euler-Poincaré equations in the continuous and discrete setting.

Consider a mechanical system determined by a Lagrangian $L : TG \longrightarrow \mathbb{R}$. It is well known that the tangent bundle TG can be left-trivialized as $TG \simeq G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a Lie group G. The motion of the mechanical system is described by applying the following principle

$$\delta \int_0^T L(g(t), \xi(t))\, dt = 0 \tag{4}$$

for all variations $\delta\xi(t)$ of the form $\delta\xi(t) = \dot{\eta}(t) + [\xi(t), \eta(t)]$, where η is an arbitrary curve on the Lie algebra with $\eta(0) = 0 = \eta(T)$ and $\delta g = g\eta$ (see [24]). This principle give rise to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \xi}\right) = \mathrm{ad}^*_\xi \left(\frac{\delta L}{\delta \xi}\right) + l^*_g \frac{\delta L}{\delta g}$$

where $\mathrm{ad}_\xi \eta = [\xi, \eta]$. If the Lagrangian L is left-invariant the above equations are written as

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \xi}\right) = \mathrm{ad}^*_\xi \left(\frac{\delta L}{\delta \xi}\right)$$

and are called the *Euler-Poincaré equations.*

3.1. **Continuous setting.**

In this subsection we deduce, from a variational principle, the Euler-Poincaré equations for Lagrangians defined on $T^{(2)}G \simeq G \times 2\mathfrak{g}$ from a left-trivization. One interesting application of this theory will be the optimal control of mechanical systems as we will seen in the next section (see [7])

Let $L : T^{(2)}G \simeq G \times 2\mathfrak{g} \to \mathbb{R}$ be a Lagrangian function, $L(g, \dot{g}, \ddot{g}) \equiv L(g, \xi, \dot{\xi})$ where $\xi = g^{-1}\dot{g}$ (left-trivialization). The problem consists on finding the critical curves of the functional

$$\mathcal{J} = \int_0^T L(g, \xi, \dot{\xi}) dt$$

among all curves satisfying the boundary conditions for arbitrary variations $\delta g = \frac{d}{d\epsilon} |_{\epsilon=0}\, g_\epsilon$, where,$\epsilon \mapsto g_\epsilon$ is a smooth curve in G such that $g_0 = g$.

We define, for any ϵ, $\xi_\epsilon := g_\epsilon^{-1}\dot{g}_\epsilon$. The corresponding variations $\delta\xi$ induced by δg are given by $\delta\xi = \dot{\eta} + [\xi, \eta]$ where $\eta := g^{-1}\delta g \in \mathfrak{g}$ $(\delta g = g\eta)$. Therefore

$$
\begin{aligned}
&\delta \int_0^T L(g(t), \xi(t), \dot{\xi}(t)) dt = \\
&\frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_0^T L(g_\epsilon(t), \xi_\epsilon(t), \dot{\xi}_\epsilon(t)) dt = \\
&\int_0^T \left(\langle \frac{\partial L}{\partial g}, \delta g \rangle + \langle \frac{\delta L}{\delta \xi}, \delta \xi \rangle + \langle \frac{\delta L}{\delta \dot{\xi}}, \delta \dot{\xi} \rangle \right) dt = \\
&\int_0^T \left(\langle \frac{\partial L}{\partial g}, \delta g \rangle + \langle \frac{\delta L}{\delta \xi}, \delta \xi \rangle + \langle \frac{\delta L}{\delta \dot{\xi}}, \frac{d}{dt}(\delta \xi) \rangle \right) dt = \\
&\int_0^T \left(\langle \frac{\partial L}{\partial g}, \delta g \rangle + \langle \frac{\delta L}{\delta \xi}, \delta \xi \rangle + \langle -\frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}}, \delta \xi \rangle \right) dt = \\
&\int_0^T \left(\langle \frac{\partial L}{\partial g}, g\eta \rangle + \langle \frac{\delta L}{\delta \xi} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}}, \frac{d}{dt}\eta + [\xi, \eta] \rangle \right) dt = \\
&\int_0^T \Big\langle \left(-\frac{d}{dt} + ad^*_\xi \right) \left(\frac{\delta L}{\delta \xi} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}} \right), \eta \Big\rangle dt + \int_0^T \Big\langle l^*_g \left(\frac{\partial L}{\partial g} \right), \eta \Big\rangle dt = 0,
\end{aligned}
$$

where we have used integration by parts and the vanishing initial and endpoint conditions $\eta(0) = \eta(T) = \dot{\eta}(0) = \dot{\eta}(T) = 0$. Thus, the stationary condition $\delta \mathfrak{J} = 0$ implies the *second-order Euler-Lagrange equations*,

$$
l^*_g \frac{\partial L}{\partial g} + \left(-\frac{d}{dt} + ad^*_\xi \right) \left(\frac{\delta L}{\delta \xi} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}} \right) = 0
$$

that is,

$$
l^*_g \frac{\partial L}{\partial g} - \frac{d}{dt}\frac{\delta L}{\delta \xi} + \frac{d^2}{dt^2}\frac{\delta L}{\delta \dot{\xi}} + ad^*_\xi \frac{\delta L}{\delta \xi} - ad^*_\xi \left(\frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}} \right) = 0. \tag{5}
$$

If the Lagrangian is invariant under an action of the Lie group, the equations of motion are

$$
\frac{d^2}{dt^2}\frac{\delta L}{\delta \dot{\xi}} - \frac{d}{dt}\frac{\delta L}{\delta \xi} + ad^*_\xi \frac{\delta L}{\delta \xi} - ad^*_\xi \left(\frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}} \right) = 0. \tag{6}
$$

These equations are called *second order Euler-Poincaré equations.*

In a recent paper [10], the authors studied invariant higher order problems and obtain the equations (6) working in a reduced Lagrangian setting on $\mathfrak{g} \times \mathfrak{g}$.

3.2. **Discrete setting.** Now, we consider the associated discrete problem. The second order tangent bundle is left-trivialized as $T^{(2)}G \simeq G \times 2\mathfrak{g}$ and then we choose its natural discretization as three copies of the Lie group (we recall that the prescribed discretization of a Lie algebra $\mathfrak{g}$ is its associated Lie group G). Consequently, we develop the discrete Euler-Lagrange equations for the discrete Lagrangians defined on $G \times G \times G = 3G$.

Let $L_d : 3G \to \mathbb{R}$ be a discrete Lagrangian where G is a finite dimensional Lie group. As in the previous section, we define $W_k = g_k^{-1} g_{k+1}$. Taking

variations for W_k, where we denote $\Sigma_k = g_k^{-1}\delta g_k$, we obtain

$$\begin{aligned}\delta W_k &= -g_k^{-1}\delta g_k g_k^{-1} g_{k+1} + g_k^{-1}\delta g_{k+1} \\ &= -\Sigma_k W_k + g_k^{-1} g_{k+1} g_{k+1}^{-1}\delta g_{k+1} \\ &= -\Sigma_k W_k + W_k \Sigma_{k+1},\end{aligned}$$

where $g_k, W_k \in G$ and $\Sigma_k \in \mathfrak{g}$.

The equations of motion are the critical paths of the discrete action

$$\sum_{k=0}^{N-2} L_d(g_k, W_k, W_{k+1})$$

with boundary conditions $\Sigma_0 = \Sigma_1 = \Sigma_{N-1} = \Sigma_N = 0$ since we are assuming that g_0, g_1, g_{N-1} and g_N fixed. Therefore, after some computations we obtain the equations

$$\begin{aligned}&l^*_{g_{k-1}} D_1 L_d(g_{k-1}, W_{k-1}, W_k) + l^*_{W_{k-1}} D_2 L_d(g_{k-1}, W_{k-1}, W_k) \\ &- r^*_{W_k} D_2 L_d(g_k, W_k, W_{k+1}) - r^*_{W_k} D_3 L_d(g_{k-1}, W_{k-1}, W_k) \\ &+ l^*_{W_{k-1}} D_3 L_d(g_{k-2}, W_{k-2}, W_{k-1}) = 0\end{aligned}$$

These equation, together with the **reconstruction equation** $W_k = g_k^{-1} g_{k+1}$, are called *discrete second order Euler-Lagrange equations .*

If L_d is G invariant in the sense that $L_d(g_k, W_{k-1}, W_k) = L_d(hg_k, W_{k-1}, W_k)$ for all $h \in G$ then we can define the reduced lagrangian $l_d : G \times G \to \mathbb{R}$ and the equations are rewritten as

$$\begin{aligned}0 &= l^*_{W_{k-1}} D_1 l_d(W_{k-1}, W_k) - r^*_{W_k} D_1 l_d(W_k, W_{k+1}) \\ &- r^*_{W_k} D_2 l_d(W_{k-1}, W_k) + l^*_{W_{k-1}} D_2 l_d(W_{k-2}, W_{k-1})\end{aligned}$$

and are called the *discrete second-order Euler-Poincaré equations.*

Remark 3.1. Is easy to extend these techniques for higher order discrete mechanics (see [2]). Consider a mechanical system determined by a Lagrangian $L : T^{(k)}G \longrightarrow \mathbb{R}$. It is well known that the tangent bundle $T^{(k)}G$ can be left-trivialized as $T^{(k)}G \simeq G \times k\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra G.

Now, we consider the associated discrete problem. First, we replace the higher order tangent bundle by $(k+1)$ copies of the group since the prescribed discretization of each $\mathfrak{g}$ is the Lie group G. At this point, we develop the discrete Euler-Poincaré equations for the discrete Lagrangians defined on $G \times kG$.

Let $L_d : G \times kG \to \mathbb{R}$ be a discrete Lagrangian where G is a finite dimensional Lie group. As before, denote by $W_i = g_i^{-1} g_{i+1}$ and $\Sigma_i = g_i^{-1}\delta g_i$. Taking variations over W_i we obtain

$$\begin{aligned}\delta W_i &= -g_i^{-1}\delta g_i \, g_i^{-1} g_{i+1} + g_i^{-1}\delta g_{i+1} \\ &= -\Sigma_i W_i + g_i^{-1} g_{i+1} g_{i+1}^{-1}\delta g_{i+1} \\ &= -\Sigma_i W_i + W_i \Sigma_{i+1},\end{aligned}$$

where $g_i, W_i \in G$ and $\Sigma_i \in \mathfrak{g}$.

The equations of motion are the critical paths of the discrete action

$$\min \sum_{i=0}^{N-k} L_d(g_i, W_{(i,i+k-1)})$$

with boundary conditions $\Sigma_0 = \ldots = \Sigma_{k-1} = 0$, $\Sigma_{N-k+1} = \ldots = \Sigma_N = 0$ and $g_0, \ldots, g_{k-1}$ and $g_{N-k+1}, \ldots, g_N$ fixed.

Taking variations we deduce

$$\begin{aligned}
&\delta \sum_{i=0}^{N-k} L_d(g_i, W_{(i,i+k-1)}) = \\
&\sum_{i=k}^{N-k} \Big[D_1 L_d(g_i, W_{(i,i+k)}) \, (g_i \Sigma_i) \\
&+ \sum_{j=2}^{k+1} D_j L_d(g_i, W_{(i,i+k-1)}) \, (-\Sigma_{j+i-2} W_{j+i-2} + W_{j+i-2} \Sigma_{j+i-1}) \Big]
\end{aligned}$$

where we denote by $W_{(i,j)} = (W_i, W_{i+1}, \ldots, W_{j-1}, W_j)$.

Therefore, the *discrete higher-order Euler-Lagrange equations* on $G \times kG$ are given by

$$\begin{aligned}
0 = {} & l^*_{g_{i-1}} D_1 L_d(g_{i-1}, W_{(i-1,i+k-1)}) \\
& + \sum_{j=2}^{k+1} \left(l^*_{W_{i-1}} \right) D_j L_d(g_{i-j+1}, W_{(i-j+1,i-j+k)}) \\
& - \sum_{j=2}^{k+1} \left(r^*_{W_i} \right) D_j L_d(g_{i-j+2}, W_{(i-j+2,i-j+k+1)}).
\end{aligned}$$

where $k \leq i \leq N - k$.

These equations, together with the reconstruction equation $W_i = g_i^{-1} g_{i+1}$ are called the *discrete higher-order Euler-Lagrange equations*. If L_d is G-invariant, that is $L_d(g_i, W_{(i,i+k-1)}) = L_d(hg_i, W_{(i,i+k-1)})$ $\forall h \in G$, we can consider the reduced Lagrangian $l_d : kG \to \mathbb{R}$. Then the *discrete higher-order Euler-Poincaré equations* on the reduced space kG are given by

$$\begin{aligned}
0 = {} & \sum_{j=2}^{k+1} \left(l^*_{W_{i-1}} \right) D_j L_d(W_{(i-j+1,i-j+k)}) \\
& - \sum_{j=2}^{k+1} \left(r^*_{W_i} \right) D_j L_d(W_{(i-j+2,i-j+k+1)}).
\end{aligned}$$

4. Discrete Optimal control problems on Lie groups

The proposal of this section is to study optimal control problems in the case of fully actuated mechanical systems. The discrete approximation to the solutions of the system have a purely discrete variational formulation and as a consequence, the integrators defined in this way are symplectic (Poisson)-momentum preserving. By using backward error analysis, it is well known that these integrators have a good energy behavior (see [25]).

As particular examples, we will study the optimal control of the rigid body and the Cosserat rod. The configuration groups in these examples are $SO(3)$ and $SE(3)$ respectively. Both are particular cases of *quadratic* Lie groups, which are defined as

$$G = \left\{Y \in GL(n, \mathbb{R}) \mid Y^T P Y = Y\right\}$$

where $P \in GL(n, \mathbb{R})$ is a given matrix (here, $GL(n, \mathbb{R})$ denotes the general linear group of degree n). The corresponding Lie algebra is

$$\mathfrak{g} = \{\Omega \in \mathfrak{g}l(n, \mathbb{R}) \mid P\Omega + \Omega P = 0\}.$$

As mentioned in subsection 2.1, the Cayley map, defined for quadratic Lie groups as

$$\operatorname{cay}(\xi) = \left(I - \frac{\xi}{2}\right)^{-1}\left(I + \frac{\xi}{2}\right),$$

where $\xi \in \mathfrak{g}$, also gives a useful and simpler discretization of these systems.

4.1. **Example: Rigid body.** The rigid body problem is very well known in the literature. This setting is deeply studied in [17, 18, 19] among other references.

The continuous equations of motion of the controlled rigid body system are the following

$$\begin{aligned} \dot{\Omega}_{(1)} &= \rho_1 \Omega_{(2)} \Omega_{(3)} + u_1, \\ \dot{\Omega}_{(2)} &= \rho_2 \Omega_{(1)} \Omega_{(3)} + u_2, \\ \dot{\Omega}_{(3)} &= \rho_3 \Omega_{(1)} \Omega_{(2)} + u_3, \end{aligned} \tag{7}$$

where $(\Omega_{(1)}, \Omega_{(2)}, \Omega_{(3)}) = \Omega$ and $(\dot{\Omega}_{(1)}, \dot{\Omega}_{(2)}, \dot{\Omega}_{(3)}) = \dot{\Omega} \in \mathbb{R}^3$, u_i are the control forces and $\rho_i \in \mathbb{R}$ are a redefinition of the inertia momenta of the problem. In the sequel we will use the typical identification of the Lie algebra of $SO(3), \mathfrak{so}(3)$ with $\mathbb{R}^3$ by $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$, that is if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$\hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

Consequently $x \times y = -[\hat{x}, \hat{y}] = ad_{\hat{x}}\hat{y}$. With some abuse of notation, we will directly identify $\mathbb{R}^3$ with $\mathfrak{so}(3)$ by omitting the hat notation.

Our fixed boundary conditions are $(R(0), \Omega(0))$ and $(R(T), \Omega(T))$, where $R(t) \in SO(3)$ is the attitude of the rigid body subject to the conditions $\dot{R} = R\Omega$ and $\delta R = R\eta$, with η an arbitrary element of $\mathfrak{so}(3)$. Besides the equations, the cost functional is

$$\mathcal{C} = \int_0^T \frac{1}{2} u^T u \, dt,$$

where $u = (u_1, u_2, u_3)$. From eqs. (7) we can work out u in terms of Ω and $\dot{\Omega}$. Consequently, we can define the function $l : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathbb{R}$ in the following way

$$l(\Omega, \dot{\Omega}) = \frac{1}{2} u^T(\Omega, \dot{\Omega}) u(\Omega, \dot{\Omega}).$$

Therefore, the Lagrangian function has the following form:

$$\begin{aligned} l(\Omega,\dot\Omega) &= \frac{1}{2}\left(\dot\Omega_{(1)}-\rho_1\Omega_{(2)}\Omega_{(3)}\right)^2+\frac{1}{2}\left(\dot\Omega_{(2)}-\rho_2\Omega_{(1)}\Omega_{(3)}\right)^2+\\ &+ \frac{1}{2}\left(\dot\Omega_{(3)}-\rho_3\Omega_{(1)}\Omega_{(2)}\right)^2. \end{aligned} \tag{8}$$

With this redefinition, the cost functional becomes

$$\mathcal{C}=\int_0^T l(\Omega,\dot\Omega)\,dt.$$

• *Discrete setting*: Our goal is to find and algorithm in N steps of time size h, i.e. $Nh=T$, that both minimizes the cost functional and respects the boundary conditions above. In order to that, we fix a discretization setting

$$R_{k+1}=R_k\,\tau(h\Omega_k),\quad \delta R_k=R_k\eta_k, \tag{9}$$

where $\eta_k\in\mathfrak{so}(3)$ such that $\eta_0=\eta_N=0$ and $\tau(h\Omega_k)\in SO(3)$ is choosen to be a general retraction map. As mentioned before, the first equation $R_{k+1}=R_k\,\tau(h\Omega_k)$ is called the reconstruction equation. From (9), it is easy to obtain the variations of the algebra elements, namely

$$\delta\Omega_k=\mathrm{d}\tau^{-1}_{h\Omega_k}(-\eta_k+\mathrm{Ad}_{\tau(h\Omega_k)}\eta_{k+1})/h, \tag{10}$$

where $\mathrm{Ad}_g\,\xi=g\,\xi\,g^{-1}$, being $\xi\in\mathfrak{g}$ and $g\in G$.

Our discretization choice enables us to work with algebra elements instead of group ones. Thus, we define the discrete function $l_d:\mathfrak{so}(3)\times\mathfrak{so}(3)\to\mathbb{R}$ like $l_d(\Omega_k,\Omega_{k+1})=hl(\Omega_k,\frac{\Omega_{k+1}-\Omega_k}{h})$, where $l(\Omega,\dot\Omega)$ is explicitly defined in (8). We have set the usual discretization for the derivative $\dot\Omega_k=\frac{\Omega_{k+1}-\Omega_k}{h}$. In consequence, let the **discrete cost functional** be

$$\mathcal{C}_d=\sum_{k=0}^{N-1}l_d(\Omega_k,\Omega_{k+1}). \tag{11}$$

Therefore, our original optimal control problem defined by l and the boundary conditions $(R(0),\Omega(0))$ and $(R(T),\Omega(T))$ have become a discrete Lagrangian problem with discrete action sum (11). Applying the Hamilton's principle, taking into account the right trivialized derivative of the retraction map defined in (2.1) and considering (10), we obtain the discrete equations of motion:

$$\begin{aligned} &\mathrm{Ad}^*_{\tau(h\Omega_{k-1})}(\mathrm{d}\tau^{-1}_{h\Omega_{k-1}})^*\left(D_1l_d(\Omega_{k-1},\Omega_k)+D_2l_d(\Omega_{k-2},\Omega_{k-1})\right)\\ &-(\mathrm{d}\tau^{-1}_{h\Omega_k})^*\left(D_1l_d(\Omega_k,\Omega_{k+1})+D_2l_d(\Omega_{k-1},\Omega_k)\right)=0,\\ &k=2,...,N-1, \end{aligned} \tag{12}$$

where D_1 and D_2 represent the partial derivative w.r.t. the first and second variables respectively.

• *Boundary conditions*: from our discretization choice $R_{k+1}=R_k\tau(h\Omega_k)$, is clear that fixing Ω_k implies constraints in the neighboring points, in this case R_{k+1} and R_k. If we allow Ω_N, that means constraints at the points R_N and R_{N+1}. Since we only consider time points up to $t=Nh$, having a constraint in the beyond-terminal configuration point R_{N+1} makes no sense.

Hence, to ensure that the effect of the terminal constraint on Ω is correctely accounted for, the set of unknown algebra points (*velocities*) must be reduced to $\Omega_{0:N-1}$. Moreover, we can set $\Omega_0 = \Omega(0)$, which reduces again, since $\Omega(0)$ is fixed, the unknown velocities to $\Omega_{1:N-1}$.

On the other hand, the boundary condition $R(T)$ is enforced by the relation $\tau^{-1}(R_N^{-1}R(T)) = 0$. Recalling that $\tau(0) = e$, this last expression just means that $R_N = R(T)$. Moreover, it is possible to translate it in terms of Ω_k such that there is no need to optimize over any of the configurations R_k. In that sense, (12) together with

$$\tau^{-1}\left(\tau(h\Omega_{N-1})^{-1}...\tau(h\Omega_0)^{-1}R_0^{-1}R(T)\right) = 0,$$

form a set of $3(N-1)$ equations (since dim $(\mathfrak{so}(3)) = 3$) for the $3(N-1)$ unknowns $\Omega_{1:N-1}$. Consequently, the optimal control problem has become a nonlinear root finding problem. From the set of velocities $\Omega_{0:N-1}$ and boundary conditions $(R(0), R(T))$, we are able to reconstruct the configuration trajectory by means of the reconstruction equation $R_{k+1} = R_k\tau(h\Omega_k)$.

• *Cayley map*: the group of rigid body rotations is represented by 3×3 matrices with orthonormal column vectors corresponding to the axes of a right-handed frame attached to the body. On the other hand, the algebra $\mathfrak{so}(3)$ is the set of 3×3 antisymmetric matrices. A $\mathfrak{so}(3)$ basis can be constructed as $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, $\hat{e}_i \in \mathfrak{so}(3)$, where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. Elements $\xi \in \mathfrak{so}(3)$ can be identified with the vector $\omega \in \mathbb{R}^3$ through $\xi = \omega^\alpha\,\hat{e}_\alpha$, or $\xi = \hat{\omega}$. Under such identification the Lie bracket coincides with the standard cross product, i.e., $\mathrm{ad}_{\hat{\omega}}\,\hat{\rho} = \omega\times\rho$, for some $\rho \in \mathbb{R}^3$. Using this identification and recalling the hat isomorphism $\hat{}$ defined above, we have

$$\mathrm{cay}(\hat{\omega}) = I_3 + \frac{4}{4+\parallel\omega\parallel^2}\left(\hat{\omega} + \frac{\hat{\omega}^2}{2}\right), \tag{13}$$

where I_3 is the 3×3 identity. The linear maps $\mathrm{d}\tau_\xi$ and $\mathrm{d}\tau_\xi^{-1}$ are expressed as the 3×3 matrices

$$\mathrm{dcay}_\omega = \frac{2}{4+\parallel\omega\parallel^2}(2I_3 + \hat{\omega}), \quad \mathrm{dcay}_\omega^{-1} = I_3 - \frac{\hat{\omega}}{2} + \frac{\omega\,\omega^T}{4}. \tag{14}$$

4.2. **Example: Cosserat rod.** This example is also known as Kirchhoff's rod. The Cosserat theory of rods is given in the Lagrangian setting. A static rod corresponds to a Lagrangian system where the energy density takes the role of the Lagrangian function.

The potential energy density is the object of most importance in rod theory. This energy density function (depending on the space curve parameter) is equivalent to the Lagrangian function of a time-dependent mechanical system, such that the static equilibrium equations of a rod correspond to the Euler-Lagrange equations of the latter.

In this subsection we develop a discrete theory for the Cosserat rod and treat the associated optimal control problem. An alternatively formulation of the discrete theory for the study of symmetries is given in [15].

The original problem is defined on the tangent bundle of the manifold $Q = SO(3)\times\mathbb{R}^3 = SE(3)$ by means of the potential energy $\mathcal{W} = \mathcal{W}^{int} + \mathcal{W}^{ext} : TQ \to \mathbb{R}$. The variables of our problem are $(R, r, \dot{R}, \dot{r})$, where both

$r, \dot{r} \in \mathbb{R}^3$, $R \in SO(3)$ and $\dot{R} \in T_R SO(3)$. If we assume that the $\mathcal{W}^{int}$ is frame independent then

$$\mathcal{W}^{int}(R, r, \dot{R}, \dot{r}) = W^{int}(R^{-1}\dot{R}, R^{-1}\dot{r}) = W^{int}(u, v),$$

where $\hat{u} = R^{-1}\dot{R} \in \mathfrak{so}(3)$ and $v = R^{-1}\dot{r} \in \mathbb{R}^3$. Therefore, our new problem is defined in the left-trivialized tangent space $SE(3) \times \mathfrak{se}(3)$ as $\mathcal{W} = W^{int}(u, v) + \mathcal{W}^{ext}(R, r)$. With some abuse of notation, let define he elements of $SE(3)$ and $\mathfrak{se}(3) = \mathfrak{so}(3) \times \mathbb{R}^3$ as

$$\Phi = (R, r) = \begin{pmatrix} R & r \\ 0_3 & 1 \end{pmatrix} \in SE(3), \;\; \phi = (u, v) = \begin{pmatrix} \hat{u} & v \\ 0_3 & 0 \end{pmatrix} \in \mathfrak{se}(3), \quad (15)$$

where 0_3 is the null 1×3 matrix (both Φ and ϕ are 4×4 matrices). Finally, the total potential energy is

$$V = \int_0^T [W^{int}(u, v) + \mathcal{W}^{ext}(R, r)] \, dt.$$

The equilibrium configurations of any static system coincide with the critical points of the potential energy. In order to obtain the equations of motion, we consider the following

$$\delta\hat{u} = [\hat{u}, \hat{\Sigma_u}] + \frac{d}{dt}\hat{\Sigma_u}, \quad \delta v = \hat{u}\Sigma_v - \hat{\Sigma_u} v + \frac{d}{dt}\Sigma_v, \quad (16)$$

where

$$\hat{\Sigma_u} = R^{-1}\delta R \in \mathfrak{so}(3), \quad \Sigma_v = R^{-1}\delta r \in \mathbb{R}^3 \quad (17)$$

are independent and satisfy the boundary conditions $\Sigma_u(0) = \Sigma_u(T) = \Sigma_v(0) = \Sigma_v(T) = 0$. It is easy to imagine that both elements form a point in $\mathfrak{se}(3)$, namely

$$\Sigma = \begin{pmatrix} \hat{\Sigma_u} & \Sigma_v \\ 0_3 & 0 \end{pmatrix}.$$

Taking variations of V, considering equations (16) and the redefinition

$$n = \frac{\partial W^{int}(u, v)}{\partial v}, \quad m = \frac{\partial W^{int}(u, v)}{\partial u} \quad (18)$$

and

$$f = \frac{\partial \mathcal{W}^{ext}(R, r)}{\partial r} \quad l = \frac{\partial \mathcal{W}^{ext}(R, r)}{\partial R}, \quad (19)$$

which we consider the control forces, we finally arrive to the equations of motion

$$\begin{aligned} &\dot{n} + n \times u + f = 0, \\ &\dot{m} + n \times v + m \times u + l = 0. \end{aligned} \quad (20)$$

For more details see [15]

The optimal control problem consists on finding a trajectory of the state variables and control inputs that minimize the cost functional

$$\mathcal{C} = \int_0^T \left(f^2 + \rho_1^2 l^2\right) dt,$$

where ρ_1 is a weight constant. The control problem is subject to the following boundary conditions $\Phi(0) = (R(0), r(0))$, $\phi(0) = (u(0), v(0))$ and $\Phi(T) = (R(T), r(T))$, $\phi(T) = (u(T), v(T))$ belonging to $SE(3) \times \mathfrak{se}(3)$.

As in the rigid body example, from eqs. (20) we can obtain an expression of f and l in terms of the other variables. Furthermore, differentiating equations (18) with respect to time, we can find out $\dot{n}$ and $\dot{m}$ in terms of $((u,v),(\dot{u},\dot{v}))$ if we assume $W^{int}(u,v)$ twice differentiable, i.e., $\begin{pmatrix} \dot{n} \\ \dot{m} \end{pmatrix} = \mathcal{H}(u,v)\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}$, where $\mathcal{H}$ is the Hessian matrix of $W^{int}(u,v)$. Now, setting the function $L : \mathfrak{se}(3)\times\mathfrak{se}(3)\to\mathbb{R}$ as $L((u,v),(\dot{u},\dot{v})) = [f((u,v),(\dot{u},\dot{v}))]^2 + \rho_1^2\,[l((u,v),(\dot{u},\dot{v}))]^2$, our problem reduces to extremize the control functional

$$\mathcal{C} = \int_0^T L((u,v),(\dot{u},\dot{v}))\,dt = \int_0^T L(\phi,\dot{\phi})\,dt, \tag{21}$$

subject to the boundary conditions above. For sake of completeness we can write down the explicit form of L, namely

$$\begin{aligned}
&L((u,v),(\dot{u},\dot{v})) = f((u,v),(\dot{u},\dot{v}))^2 + \rho_1^2 l((u,v),(\dot{u},\dot{v}))^2 = \\
&\left(\mathcal{H}_{11}(u,v)\,\dot{u} + \mathcal{H}_{12}(u,v)\,\dot{v} + \partial_v W^{int}(u,v)\times u\right)^2 + \\
&+\rho_1^2(\mathcal{H}_{21}(u,v)\,\dot{u} + \mathcal{H}_{22}(u,v)\,\dot{v} + \\
&+\partial_v W^{int}(u,v)\times v + \partial_u W^{int}(u,v)\times u)^2.
\end{aligned}$$

• *Discrete Setting*: again we look for an algorithm minimizing the cost functional (21) and subject to the boundary conditions. Firstly, we define the discrete Lagrangian function $L_d : \mathfrak{se}(3)\times\mathfrak{se}(3)\longrightarrow\mathbb{R}$ as

$$L_d(\phi_k,\phi_{k+1}) = hL\left(\phi_k, \frac{\phi_{k+1}-\phi_k}{h}\right)$$

and then the **discrete cost functional**

$$\mathcal{C}_d = \sum_{k=0}^{N-1} L_d(\phi_k,\phi_{k+1}). \tag{22}$$

From now on, our discussion is equivalent to the rigid body example developed in (4.1). We fix the discretization setting

$$\Phi_{k+1} = \Phi_k\tau(h\phi_k), \quad \delta\,\Phi_k = \Phi_k\Sigma_k, \tag{23}$$

where $\Sigma_k \in \mathfrak{se}(3)$ s.t. $\Sigma_0 = \Sigma_N = 0$ and $\tau : \mathfrak{se}(3)\to SE(3)$ is a general retraction map. Consequently, the variations of ϕ_k are

$$\delta\phi_k = \mathrm{d}\tau^{-1}_{h\phi_k}(-\Sigma_k + \mathrm{Ad}_{\tau(h\phi_k)}\Sigma_{k+1})/h,$$

and the discrete equations of motion:

$$\begin{aligned}
&\mathrm{Ad}^*_{\tau(h\phi_{k-1})}(\mathrm{d}\tau^{-1}_{h\phi_{k-1}})^*\left(D_1L_d(\phi_{k-1},\phi_k) + D_2L_d(\phi_{k-2},\phi_{k-1})\right) \\
&-(\mathrm{d}\tau^{-1}_{h\phi_k})^*\left(D_1L_d(\phi_k,\phi_{k+1}) + D_2L_d(\phi_{k-1},\phi_k)\right) = 0, \\
&\\
&k = 2,...,N-1,
\end{aligned}$$

• *Boundary conditions*: our reconstruction equation $\Phi_{k+1} = \Phi_k\tau(h\phi_k)$ and boundary conditions $(\Phi(0),\phi(0))$, $(\Phi(0),\phi(0))$ reduce our set of unknowns to $\phi_{1:N-1}$. The discrete equations of motion together with the

boundary condition $\Phi(T) = \Phi_N$ enforced by the equation

$$\tau^{-1}\left(\tau(h\phi_{N-1})^{-1}...\tau(h\phi_0)^{-1}\Phi_0^{-1}\Phi(T)\right) = 0,$$

where Φ^{-1} is given by

$$\Phi^{-1} = \begin{pmatrix} R^{-1} & -R^{-1}r \\ 0_3 & 1 \end{pmatrix},$$

form a set of $6(N-1)$ equations for the $6(N-1)$ unknowns $\phi_{1:N-1}$ (since $\dim(\mathfrak{se}(3)) = 6$). Again, the optimal control problem has become a nonlinear root finding problem.

- *Cayley map*: considering the elements of $SE(3)$ and $\mathfrak{se}(3)$ defined in (15), the Cayley transform $\mathrm{cay} : \mathfrak{se}(3) \to SE(3)$ is defined by

$$\mathrm{cay}(\phi) = \begin{pmatrix} \mathrm{cay}_{SO(3)}(\hat{u}) & \mathrm{dcay}_u\, v \\ 0_3 & 1 \end{pmatrix}, \tag{24}$$

where $\mathrm{cay}_{SO(3)} : \mathfrak{so}(3) \to SO(3)$ is given by (13) and $\mathrm{dcay} : \mathbb{R}^3 \to \mathbb{R}^3$ by (14).

4.2.1. *A direct computation.* Choosing $\tau = \mathrm{cay}$ in (23) and taking into account (24), the reconstruction equation $\Phi_{k+1} = \Phi_k \mathrm{cay}(h\phi_k)$ splits as follows:

$$R_{k+1} = R_k \mathrm{cay}_{SO(3)}(h\hat{u}_k), \quad r_{k+1} = r_k + hR_k \mathrm{dcay}_{hu_k}(v_k).$$

For sake of simplicity, we take a truncation of the second equation such that the **reconstruction setting** stands as

$$R_{k+1} = R_k \mathrm{cay}_{SO(3)}(hu_k), \quad r_{k+1} = r_k + hR_k v_k. \tag{25}$$

The second equation in (25) clearly represents the easiest discretization of the frame independence condition $v = R^{-1}r$, which in our opinion makes the truncation non-trivial. In order to complete the discrete setting, we define $g_k = \mathrm{cay}_{SO(3)}(h\hat{u}_k)$ and the variations of the $SE(3)$ elements as

$$\delta R_k = R_k(\widehat{\Sigma_u})_k, \quad \delta r_k = R_k(\Sigma_v)_k, \tag{26}$$

such that $(\widehat{\Sigma_u})_0 = (\widehat{\Sigma_u})_N = 0_{3\times 3}$, $(\Sigma_v)_0 = (\Sigma_v)_N = 0$.

By means of (25) and (26) we can completely determine δu_k and δv_k in terms of u_k, v_k, $(\Sigma_u)_k$ and $(\Sigma_v)_k$:

$$\begin{aligned}\delta u_k &= \frac{1}{h}\Big[\mathrm{Ad}_{g_k}(\Sigma_u)_{k+1} - (\Sigma_u)_k + \frac{h}{2}\mathrm{ad}_{\hat{u}_k}(\Sigma_u)_k - \frac{h}{2}\mathrm{ad}_{\hat{u}_k}\mathrm{Ad}_{g_k}(\Sigma_u)_k \\ &+ \frac{h^2}{4}\hat{u}_k(\Sigma_u)_k\hat{u}_k - \frac{h^2}{4}\hat{u}_k\left(\mathrm{Ad}_{g_k}(\Sigma_u)_{k+1}\right)\hat{u}_k\Big], \\ \delta v_k &= -(\widehat{\Sigma_u})_k v_k + \frac{1}{h}g_k(\Sigma_v)_{k+1} - \frac{1}{h}(\Sigma_v)_k.\end{aligned}$$

Taking variations of $\mathcal{C}_d$ in (22) and after long calculations, we arrive to the following algorithm:

$$\begin{aligned}
&\mathrm{Ad}^*_{g_{k-1}}\Upsilon^{SO(3)}_{(k-2,k-1,k)} - \Upsilon^{SO(3)}_{(k-1,k,k+1)} + \\
&+\frac{h}{2}\mathrm{ad}^*_{\hat{u}_k}\Upsilon^{SO(3)}_{(k-1,k,k+1)} - \frac{h}{2}\mathrm{Ad}^*_{g_{k-1}}\mathrm{ad}^*_{g_{k-1}}\Upsilon^{SO(3)}_{(k-2,k-1,k)} + \\
&+\frac{h^2}{4}\hat{u}^*_k\Upsilon^{SO(3)}_{(k-1,k,k+1)}\hat{u}^*_k - \frac{h^2}{4}\mathrm{Ad}^*_{g_{k-1}}\hat{u}^*_{k-1}\Upsilon^{SO(3)}_{(k-2,k-1,k)}\hat{u}^*_{k-1} + \\
&-h[\Upsilon^{\mathbb{R}^3}_{(k-1,k,k+1)}, v_k] = 0, \qquad (27)
\end{aligned}$$

$$g^T_{k-1}\Upsilon^{\mathbb{R}^3}_{(k-2,k-1,k)} - \Upsilon^{\mathbb{R}^3}_{(k-1,k,k+1)} = 0, \qquad k = 2, ..., N-2.$$

$$\begin{aligned}
R_{k+1} &= R_k \mathrm{cay}_{SO(3)}(h\hat{u}_k), && k = 0, ..., N-1 \\
r_{k+1} &= r_k + hR_k v_k, && k = 0, ..., N-1.
\end{aligned} \qquad (28)$$

Here $\Upsilon^{SO(3)} \in \mathfrak{so}^*(3)$ and $\Upsilon^{\mathbb{R}^3} \in \mathbb{R}^3$, stands for

$$\begin{aligned}
\Upsilon^{SO(3)}_{(a,b,c)} &:= D_1L_d(u_b, v_b, u_c, v_c) + D_3L_d(u_a, v_a, u_b, v_b), \\
\Upsilon^{\mathbb{R}^3}_{(a,b,c)} &:= D_2L_d(u_b, v_b, u_c, v_c) + D_4L_d(u_a, v_a, u_b, v_b),
\end{aligned}$$

being (a, b, c) integers from 2 to $N-2$. Both operators Ad^* and ad^* act over elements of $\mathfrak{so}(3)^*$. The dual algebra element $\xi^*\omega\xi^* \in \mathfrak{so}(3)^*$ is defined such that $\langle\xi^*\omega\xi^*, \eta\rangle = \langle\omega, \xi\eta\xi\rangle$ for $\omega \in \mathfrak{so}(3)^*$, $\xi, \eta \in \mathfrak{so}(3)$ and $\langle\cdot,\cdot\rangle$ the natural pairing between $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$.

Finally, we have obtained an algorithm that approximates in an implicit and non linear way the solution of the optimal control problem for the Cosserat rod setting.

5. Conclusions and Future works

5.1. **Conclusions.** In this paper, we have designed new variational integrators for optimal control of mechanical systems showing how developments in the theory of discrete mechanics and variational methods [25] can be used to construct numerical optimal control algorithms with certain desirable features. The methods are available for developing integrators on higher-order problems. The main idea is to use discrete variational calculus on Lie groups using the Lagrange-d'Alembert principle and to derive the discrete Euler-Poincaré equation for discrete Lagrangians corresponding to a discretization of the second order Lagrangian defined on the trivialized space (left-trivialized) $G \times 2\mathfrak{g}$.

It is also possible to use our techniques and the numeric integrator obtained in this paper for other interesting problems, like for instance the theory of k-splines on $SO(3)$ [10], [26]. In this paper, we show two applications of second-order mechanics on the Lie groups on $SO(3)$ and $SE(3)$, the rigid body and the Cosserat rod, respectively.

5.2. **Future Work.** A complete study of symmetry reduction, discrete hamiltonian description, preservation of geometric structure and numerical simulations will be developed in a future paper. This discrete approach will be studied and adapted to the Lie groupoid setting [6], [14], [21]. One interesting point, for future work, is to extend our methods to underactuated constraints systems using discrete constrained variational calculus (see [7] for the continuous counterpart). The case of optimal control problems for mechanical systems with nonholonomic constraints will be also studied using some of the ideas exposed along the paper [12].

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